NPTEL COURSE - Introduction to Commutative Algebra

Assignment - 4

1. Is $\oplus_{i=1}^{\infty}\mathbb{Z}_{n_i}$, where $n_i \in \mathbb{N}$ a free $\mathbb{Z}$ module ?

   Solution: Suppose $\oplus_{i=1}^{\infty}\mathbb{Z}_{n_i}$ is a free $\mathbb{Z}$ module, i.e., there exists a $\mathbb{Z}$-module isomorphism $\phi : \oplus_{i=1}^{\infty}\mathbb{Z}_{n_i} \to \oplus_{i\in I}\mathbb{Z}$. Let $e_1 \in \oplus_{i=1}^{\infty}\mathbb{Z}_{n_i}$ be the vector with $\bar{1}$ at the first coordinate and $\bar{0}$ everywhere else. Then e_1 is an element of order n_1 and hence the order of $\phi(e_1)$ divides n_1. Hence $\phi(e_1) = 0$. This is a contradiction to the assumption that ϕ is an isomorphism. Therefore, ϕ can not be an isomorphism, i.e., $\oplus_{i=1}^{\infty}\mathbb{Z}_{n_i}$ is not a free module.

2. Let M be a finitely generated A module and $\phi : M \to A^n$ be a surjective homomorphism.

   (a) If $\phi(m_i) = e_i$ for $i = 1, \dots, n$, then prove that $M = \langle m_1, \dots, m_n\rangle \oplus \ker\phi$.

   (b) Deduce that $\ker\phi$ is a finitely A-submodule of M.

   Solution: (a) Since $\langle m_1, \dots, m_n\rangle$ and $\ker\phi$ are submodule of M, $\langle m_1, \dots, m_n\rangle \oplus \ker\phi$ is submodule of M. Let $m \in M$. Therefore $\phi(m) \in A^n$. We have

$$\begin{aligned}\phi(m) &= \lambda_1 e_1 + \cdots + \lambda_n e_n, \text{ where } \lambda_i \in A\\ &= \lambda_1\phi(m_1) + \cdots + \lambda_n\phi(m_n)\\ &= \phi(\lambda_1 m_1 + \cdots + \lambda_n m_n).\end{aligned}$$

   and $\phi(m - \lambda_1 m_1 + \cdots + \lambda_n m_n) = 0$. Therefore $m \in \langle m_1, \dots, m_n\rangle + \ker\phi$. We can observe that $\langle m_1, \dots, m_n\rangle \cap \ker\phi = 0$. Hence M is a submodule of $\langle m_1, \dots, m_n\rangle \oplus \ker\phi$.

   (b) We have an isomorphism

$$\frac{\ker\phi + \langle m_1, \dots, m_n\rangle}{\langle m_1, \dots, m_n\rangle} \cong \frac{\ker\phi}{\langle m_1, \dots, m_n\rangle \cap \ker\phi}$$

   Note that $\frac{\ker\phi + \langle m_1, \dots, m_n\rangle}{\langle m_1, \dots, m_n\rangle}$ is finitely generated and $\langle m_1, \dots, m_n\rangle \cap \ker\phi = 0$. Therefore $\ker\phi$ is finitely generated.

3. Let $R = k[x^2, xy, xy^2, xy^3, \ldots]$ be polynomial ring over a field k. Prove that $I = \langle x^2, xy, xy^2, xy^3, \ldots \rangle$ is not finitely generated ideal over R.

Solution: Suppose I is finitely generated, say generated by $\{f_1, \ldots, f_r\}$. It is clear that there exist a positive integer $n > 0$ such that each $f_i \in k[x^2, xy, xy^2, xy^3, \ldots, xy^n]$, for all $i = 1, \ldots, r$. Since $xy^{n+1} \in I$, we can write $xy^{n+1} = \sum_{j=1}^{r_1} \alpha_j f_{i_j}$ for some $\alpha_j \in R$ and $\{f_{i_1}, \ldots, f_{i_{r_1}}\} \subseteq \{f_1, \ldots, f_r\}$.

Let $S = k[x, y]$. Considering the above expression for xy^{n+1} in S, and setting $f_{i_j} = xg_{i_j}$ for some $g_{i_j} \in S$, we get

$$y^{n+1} = \sum_{j=1}^{r_1} \alpha_j g_{i_j}.$$

Now putting $x = 0$, we get that

$$y^{n+1} = \sum_{j=1}^{r_1} \beta_j g'_{i_j},$$

where β_j's are either 0 or in k (since each monomial in α_j is a multiple of x, except constant term) and g'_{i_j} contains only powers of y. Note also that the maximum power of y appearing in the terms of f_{i_j}'s is n. Therefore, the above equation contradicts the linear independence of the set $\{1, y, \ldots, y^{n+1}\}$ over k. This contradiction says that our assumption that I is finitely generated is wrong. Therefore I is not finitely generated ideal in R.